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The Lower and Upper Forcing Edge-to-vertex Geodetic Numbers of a Graph

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Abstract: *A subset $T \subseteq S$ is called a forcing subset for S if S is the unique minimum edge-to-vertex geodetic set containing T. A forcing subset for S of minimum cardinality is a minimum forcing subset of S. The forcing edge-to-vertex geodetic number of S, denoted by $f_{ev}(S)$, is the cardinality of a minimum forcing subset of S. The lower forcing edge-to-vertex geodetic number of G, denoted by $f_{ev}(G)$, is $f_{ev}(G) = min\{f_{ev}(S)\}$, where the minimum is taken over all minimum edge-to-vertex geodetic sets S in G. The upper forcing edge-to-vertex geodetic number of G, denoted by $f^{+}_{ev}(G)$, is $f^{+}_{ev}(G) = max\{f_{ev}(S)\}$, where the maximum is taken over all minimum edge-to-vertex geodetic sets S in G. These concepts were studied in [3], [4] and [9]. In this paper, we extend the study of lower and upper forcing edge-to-vertex geodetic numbers of graphs whose minimum edge-to-vertex geodetic sets containing antipodal edges.*

Keywords: *edge-to-vertex geodetic number, lower forcing edge-to-vertex geodetic number, upper forcing edge-to-vertex geodetic number.*

AMS Subject Classification: 05C12.

1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by p and q respectively. For basic definitions and terminologies we refer to [1]. For vertices u and v in a connected graph G, the *distance* $d(u, v)$ is the length of a shortest $u - v$ path in G. A $u - v$ path of length $d(u, v)$ is called a $u - v$ *geodesic*. A vertex x is said to *lie* on a $u - v$ geodesic if x is a vertex of a $u - v$ geodesic.

A *geodetic set* of G is a set $S \subseteq V(G)$ such that every vertex of G is contained in a geodesic joining a pair of vertices of S. The *geodetic number* $g(G)$ of G is the minimum cardinality of a geodetic set and any geodetic set of cardinality $g(G)$ is called a *geodetic basis* or simply a *g-set* of G. A set $S \subseteq E(G)$ is called an *edge-to-vertex geodetic set* if every vertex of G is either incident with an edge of S or lies on a geodesic joining a pair of edges of S. The *edge-to-vertex geodetic number* $g_{ev}(G)$ of G is the minimum cardinality of its edge-to-vertex geodetic sets and any edge-to-vertex geodetic set of cardinality $g_{ev}(G)$ is called an *edge-to-vertex geodetic basis* of G or a $g_{ev}(G)$-set of G. This concept is studied in [8].

For any edge e of a connected graph G, the *edge-to-edge eccentricity* $e_3(e)$ of e is $e_3(e) = max\{d(e, f): f \in E(G)\}$. The minimum eccentricity among the edges of G is the *edge-to-edge radius*, *rad G* and the maximum eccentricity among the edges of G is the *edge-to-edge diameter*, *diam G* of G. Two edges e and f are *antipodal* if $d(e, f) = diam\ G$ or $d(G)$. This concept was studied in [7]. A vertex v is an *extreme vertex* of a graph G if the subgraph induced by its neighbors is complete. An edge e of a graph G is called an *extreme edge* of G if one of its end is an extreme vertex of G.

A *wheel graph* is a graph formed by connecting a single vertex to all vertices of a cycle and it is denoted by W_p. In this paper, W_p denotes a wheel graph with p+1 vertices ($p \geq 3$) which is formed by connecting a single vertex to all vertices of a cycle of length p. The wheel graph has diameter two if $p > 3$ and one if $p = 3$. The graph $C_4 \times K_2$ is often denoted by Q_3 and is called 3-*cube*. The *triangular snake* TS_n is obtained from the path P_n by replacing every edge of a path by a triangle C_3. The *square* *of graph G* denoted by G^2 is defined to be the graph with the same vertex set as G and in which two vertices u and v are joined by an edge in $G \Leftrightarrow 1 \le d(u,v) \le 2$. These concept were studied in [2],[5] and [6].

A subset $T \subseteq S$ is called a *forcing subset* for S if S is the unique minimum edge-to-vertex geodetic set containing T. A forcing subset for S of minimum cardinality is a *minimum forcing subset* of S. The *forcing edge-to-vertex geodetic number* of S, denoted by $f_{ev}(S)$, is the cardinality of a minimum forcing subset of S. The lower forcing edge-to-vertex geodetic number of G, denoted by $f_{ev}(G)$, is $f_{ev}(G) = \min\{f_{ev}(S)\}$, where the minimum is taken over all minimum edge-to-vertex geodetic sets S in G. The *upper forcing edge-to-vertex geodetic number* of G, denoted by $f^+_{ev}(G)$, is $f^+_{ev}(G) = \max\{f_{ev}(S)\}$, where the maximum is taken over all minimum edge-to-vertex geodetic sets S in G. These concepts were studied in [3],[4] and [9]. In this paper we study some more properties about the lower and upper forcing edge-to-vertex geodetic numbers in minimum edge-to-vertex geodetic sets of connected graphs.

Throughout the following G denotes a connected graph with at least three vertices. The following Theorems are used in the sequel.

Theorem 1.1.[9] For every connected graph G, $0 \le f^+_{ev}(G) \le g_{ev}(G)$.

Theorem 1.2. [3] For every connected graph G, $0 \le f_{ev}(G) \le g_{ev}(G)$.

Theorem 1.3. [3] Let G be a connected graph. Then

a) $f_{ev}(G) = 0$ if and only if G has a unique minimum edge-to-vertex geodetic set.

b) $f_{ev}(G) = 1$ if and only if G has at least two minimum edge-to-vertex geodetic sets, one of which is a unique minimum edge-to-vertex geodetic set containing one of its elements, and

c) $f_{ev}(G) = g_{ev}(G)$ if and only if no minimum edge-to-vertex geodetic set of G is the unique minimum edge-to-vertex geodetic set containing any of its proper subsets.

Theorem 1.4. [9] Let G be a connected graph. Then

a) $f^+_{ev}(G) = 0$ if and only if G has a unique minimum edge-to-vertex geodetic set.

b) $f^+_{ev}(G) = 1$ if and only if G has at least two minimum edge-to-vertex geodetic sets, in which one element of each minimum edge-to-vertex geodetic set of G does not belong to any other minimum edge-to-vertex geodetic set of G.

c) $f^+_{ev}(G) = g_{ev}(G)$ if and only if there exists at least one minimum edge-to-vertex geodetic set of G which does not contain any proper forcing subsets.

2. The Lower and Upper Forcing Edge-to-Vertex Geodetic Numbers of a Graph

The following Lemma gives the bound for the lower and upper forcing edge-to-vertex geodetic numbers of a graph. It is the extension of the results proved in [3] and [9].

Lemma 2.1. For every connected graph G, $0 \le f_{ev}(G) \le f^+_{ev}(G) \le g_{ev}(G)$.

Proof. In [3] and [9], we proved the results $0 \le f_{ev}(G) \le g_{ev}(G)$ and $0 \le f^+_{ev}(G) \le g_{ev}(G)$. Therefore, to conclude the lemma we need to prove $f_{ev}(G) \le f^+_{ev}(G)$. By the definition of the lower and upper forcing edge-to-vertex geodetic number, we see that $f^+_{ev}(G)$ is the maximum of all forcing edge-to-vertex geodetic numbers of the minimum edge-to-vertex geodetic sets, and $f_{ev}(G)$ is the minimum of all forcing edge-to-vertex geodetic numbers of the minimum edge-to-vertex geodetic sets. Hence the inequality is obvious.

Example 2.2. The bounds in Lemma 2.1 are sharp. Consider the non-trivial tree $G = T$. Since the tree has a unique minimum edge-to-vertex geodetic set, and by Theorem 1.4(a) and Theorem 1.3(a), $f^+_{ev}(G) = 0$, $f_{ev}(G) = 0$ so that $0 = f_{ev}(G) = f^+_{ev}(G)$. Also, in [3] and [9], for an even cycle C_{2p} (p = 2,3…), $f^+_{ev}(G) = 1 = f_{ev}(G)$ and $g_{ev}(G) = 2$. Moreover, all the inequalities in Lemma 2.1 are strict.

For the wheel graph $G = W_{10}$, $f^{+}_{ev}(G) = 3$, $f_{ev}(G) = 2$ and $g_{ev}(G) = 4$. Hence $0 < f_{ev}(G) < f^{+}_{ev}(G) < g_{ev}(G)$.

In [3] and [9], we showed the result, $f^{+}_{ev}(G) = 0$ if and only if G has a unique minimum edge-to-vertex set and also, $f_{ev}(G) = 0$ if and only if G has a unique minimum edge-to-vertex set. The following lemma is an extension of that result.

Lemma 2.3. For a connected graph G, $f_{ev}(G) = 0$ if and only if $f^{+}_{ev}(G) = 0$.

Proof. The proof is obvious.

Theorem 2.4. For a 3-cube graph $G = Q_3$, a set $S \subseteq E(G)$ is a minimum edge-to-vertex geodetic set if and only if S consists of a pair of antipodal edges.

Proof. Let the vertices of Q_3 be $v_1, v_2, v_3, ..., v_8$. Without loss of generality, we take v_1, v_2, v_3 and v_4 are the vertices of the outer square and v_5, v_6, v_7 and v_8 are the vertices of the inner square of the 3-cube. Then the edges v_1v_5 and v_3v_7 are a pair of antipodal edges. Let $S = \{v_1v_5, v_3v_7\}$. Clearly, S is a minimum edge-to-vertex geodetic set of Q_3. Conversely, let S be a minimum edge-to-vertex geodetic set of Q_3. Then $g_{ev}(Q_3) = |S|$. Let S' be any set of pair of antipodal edges of Q_3. Then as in the first part of this theorem, S' is a minimum edge-to-vertex geodetic set of Q_3. Hence $|S'| = |S|$. Thus $S = \{uv, xy\}$. If uv and xy are not antipodal, then any vertex that is not on the $uv - xy$ geodesic does not lie on the $uv - xy$ geodesic. Thus S is not a minimum edge-to-vertex geodetic set, which is a contradiction.

Theorem 2.5. For a 3-cube graph $G = Q_3$, $f_{ev}(G) = f^{+}_{ev}(G) = 1$.

Proof. By Theorem 2.4**,** every minimum edge-to-vertex geodetic set of Q_3 consists of pair of antipodal edges. Hence Q_3 has two independent minimum edge-to-vertex geodetic sets and it is clear that each singleton set is the minimum forcing subset for exactly one minimum edge-to-vertex geodetic set of Q_3. Hence it follows from Theorem 1.3 (a) and (b) that $f_{ev}(Q_3) =1$. Also, from Theorem 1.4 (a) and (b) that $f^{+}_{ev}(Q_3) = 1$.Thus $f_{ev}(G) = f^{+}_{ev}(G) = 1$.

Theorem 2.6. Let G be a connected graph with at least two g_{ev}–sets. If every minimum edge-to-vertex geodetic sets of G containing antipodal edges, then $f_{ev}(G) = 1 = f^{+}_{ev}(G)$.**Proof.** Let G be a connected graph. Suppose S_i, $i= 1,2,\ldots$are a collection of minimum edge-to-vertex geodetic sets containing antipodal edges of G. Since each S_i contains antipodal edges, we observe that every minimum edge-to-vertex geodetic set is independent of others. Therefore, each singleton set is the minimum forcing subset for exactly one minimum edge-to-vertex geodetic set of G. Hence, by Theorem 1.3 (a) and (b), we get $f_{ev}(G) = 1$. Also, from Theorem 1.4 (a) and (b), $f^{+}_{ev}(S_i) = 1$ for all $i= 1,2,\ldots$So that $f^{+}_{ev}(G) = 1$.Thus $f_{ev}(G) =1 = f^{+}_{ev}(G)$.

The following theorem is the interpretation of the previous theorem.

Theorem 2.7. Let G be a connected graph with at least two g_{ev}–sets. If pairwise intersection of distinct minimum edge-to-vertex geodetic sets of G is empty, then $f_{ev}(G) =1$ and $f^{+}_{ev}(G) =1$.

Proof. Given that G has at least two minimum g_{ev}–sets and for every minimum edge-to-vertex geodetic set S_i ,$i= 1,2,\ldots$such that $S_i \cap S_j = \emptyset$. Therefore S_i has an edge uv such that $uv \notin S_j$ for every minimum edge-to-vertex geodetic S_j different from S_i. Hence we obtain $f_{ev}(G) =1$. Since this is true for all minimum edge-to-vertex geodetic sets, we get $f^{+}_{ev}(G) =1$.

The following table shows that the lower and upper forcing edge-to-vertex geodetic numbers of some wheel graphs.

Table 1

Graph	$f_{ev}(G)$	$f^{+}_{ev}(G)$
W_3	1	1
W_4	2	2
W_5	1	1
W_6	1	1
W_7	2	3
W_8	2	2
W_9	1	1
W_{10}	2	3

Corollary 2.8. For a wheel graph, $f^+_{ev}(W_{3p}) = f_{ev}(W_{3p}) = 1$ if $p > 1$

Proof. Let the vertices of W_{3p} ($p > 1$) be $\{v_1, v_2, v_3, ..., v_{3p+1}\}$. Note that the minimum edge-to-vertex geodetic sets of W_{3p} are $S_1 = \{v_1v_2, v_4v_5, \ldots, v_{3p-2}v_{3p-1}\}$, $S_2 = \{v_2v_3, v_5v_6, \ldots, v_{3p-1}v_{3p}\}$, $S_3 = \{v_3v_4, v_6v_7, \ldots v_{3p}v_1\}$. It is clear that $S_1 \cap S_2 = S_2 \cap S_3 = S_1 \cap S_3 = \emptyset$. That is, pair wise intersection of minimum edge-to-vertex geodetic sets of W_{3p} is empty. Hence by theorem 2.7, we have $f_{ev}(W_{3p}) = 1$. Since this is true for all minimum edge-to-vertex geodetic sets of W_{3p}, we get $f^+_{ev}(W_{3p}) = 1$.

Theorem 2.9. For a triangular snake, $G = TS_n$ of path $P_n (n > 2)$, $f_{ev}(G) = f^+_{ev}(G) = 1$.

Proof. Let $G = TS_n$ be a triangular snake obtained from the path P_n. Consider the vertices of TS_n are $\{v_1, v_2, v_3, ..., v_n, u_1, u_2, u_3, ..., u_{n-1}\}$. The graph $G = TS_n$ is shown in Figure 1.We can easily observe that $S_1 = \{u_1v_1, u_2v_3, u_3v_4, \ldots, u_{n-1}v_n\}$ and $S_2 = \{u_1v_1, u_2v_2, u_3v_3, \ldots, u_{n-2}v_{n-2}, u_{n-1}v_n\}$ are the only two minimum g_{ev}–sets of G, and some singleton sets are minimum forcing subsets for exactly one minimum g_{ev}–set of G . Hence, by Theorem 1.3 (*a*) and (*b*), we get $f_{ev}(G) = 1$. Also, by Theorem 1.4 (*a*) and (*b*), $f^+_{ev}(S_i) = 1$ for all $i = 1, 2$. So that $f^+_{ev}(G) = 1$.Thus $f_{ev}(G) = 1 = f^+_{ev}(G)$.

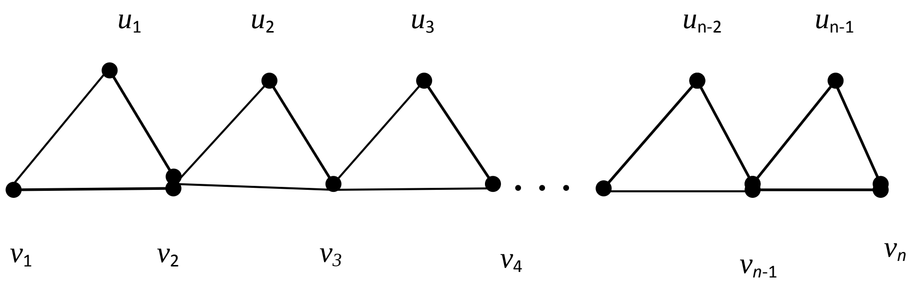

Figure 1.
Triangular snake TS_n of path P_n

Theorem 2.10. For a square path $G = p_n^2$, $f_{ev}(G) = f^+_{ev}(G) = \begin{cases} 0 \text{ if } n \text{ is even} \\ 2 \text{ if } n \text{ is odd} \end{cases}$.

Proof.

Let the vertices of the square path p_n^2 be $\{v_1, v_2, v_3, ..., v_n\}$. The graph $G = p_n^2$ is shown in Figure 2.

Case (i): *n* even & *n* > 2.

It is clear from the vertices of the square path p_n^2, $S = \{v_1v_2, v_{n-1}v_n\}$ is a unique minimum edge-to-vertex geodetic set of G. Since a square path, $1 \le d(u,v) \le 2$ for all u,v in G, we have every vertices of p_n^2 is either incident or lies on a geodesic joining of v_1v_2 and $v_{n-1}v_n$. Hence, by Theorem 1.3 (*a*), $f_{ev}(G) = 0$. Also, by Theorem 1.4 (*a*), $f^+_{ev}(G) = 0$. Thus $f_{ev}(G) = 0 = f^+_{ev}(G)$.

Case (ii): *n* odd & *n* > 3.

Since G has more than one minimum edge-to-vertex geodetic sets, and by Theorem 1.3 (a) $f_{ev}(G) \neq 0$ and by theorem 1.4(a) $f^+_{ev}(G) \neq 0$. It is clear that, the sets $S_1 = \{v_1v_2, v_{n-2}\ v_{n-1}, v_{n-1}v_n\}$, $S_2 = \{v_1v_2, v_{n-2}v_{n-1}, v_{n-2}v_n\}$, $S_3 = \{v_1v_3, v_2v_3, v_{n-1}v_n\}$, $S_4 = \{v_1v_2, v_2v_3, v_{n-1}v_n\}$, $S_5 = \{v_1v_2, v_{n-2}v_n, v_{n-1}v_n\}$, $S_6 = \{v_1v_2, v_1v_3, v_{n-1}v_n\}$ are the only minimum edge-to-vertex geodetic sets of p_n^2. It is easily verified that, each singleton set is a subset of more than one minimum edge-to-vertex geodetic sets $S_i (1 \le i \le 6)$ and hence $f_{ev}(G) \neq 1$ .Since S_3 is the unique minimum edge-to-vertex geodetic set containing $T = \{v_1v_3, v_2v_3\}$, it follows that $f_{ev}(S_3) = 2$. Hence $f_{ev}(G) = 2$. But it is easily verified that the every two element subsets of S_i are not contained in more than one minimum edge-to-vertex geodetic set $S_i (1 \le i \le 6)$ so that $f_{ev}(S_i) = 2$ for all $(1 \le i \le 6)$ and hence $f^+_{ev}(G) = 2$.

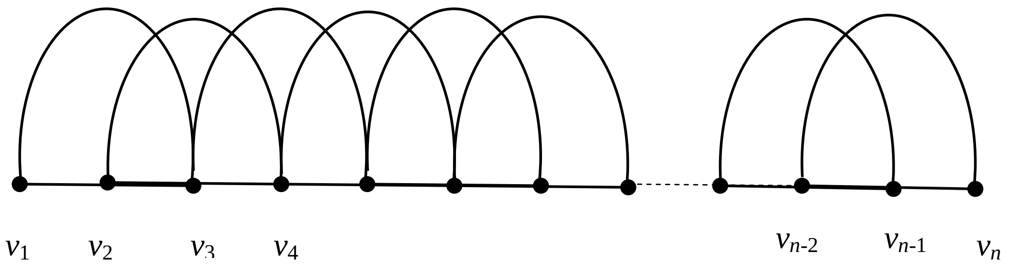

Figure 2.
Square path p_n^2

3. CONCLUSION

In this paper we considered the lower and upper forcing edge-to-vertex geodetic numbers of some graphs. One can go for the general result on $f_{ev}(G)$ and $f^+_{ev}(G)$, for every pair a, b of integers with $0 \le a \le b \le c$, there exists a connected graph G such that $f_{ev}(G) = a$, $f^+_{ev}(G)=b$ and $g_{ev}(G) = c$.

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